Equivalences of Synchronous Block Diagrams: the Case of Delays and Local Time Scales

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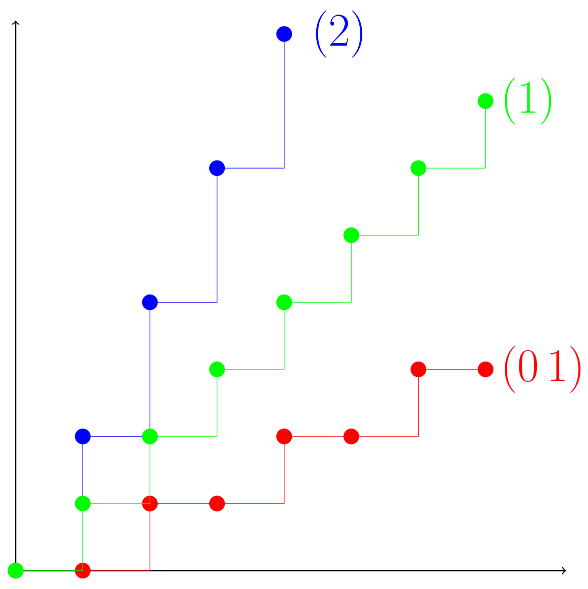
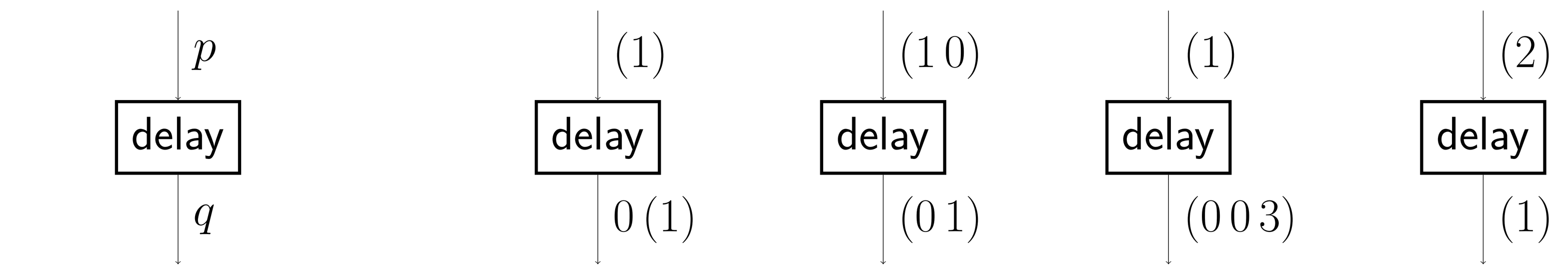
Time Warps

- **Time warps** generalize the notion of clock present in Lustre-like languages, making it possible to describe streams that grow by more than one element per tick.
- Formally, time warps are sup-preserving functions from $\omega + 1$ to itself. In other words, a monotonic function $p : \omega + 1 \to \omega + 1$ is a time warp iff
$$p(0) = 0 \text{ and } p(\omega) = \min_{n<\omega} p(n).$$
- Time warps $|\mathcal{W}|$ can be made into a partially-ordered monoid $\mathcal{W} \triangleq (|\mathcal{W}|, *, id, \leq)$ with
$$p * q \triangleq n \mapsto q(p(n)) \text{ and } p \leq q \triangleq \forall n, p(n) \leq q(n).$$
- In practice, one may want to consider an algorithmically-tractable subclass of time warps. For instance, those presentable as running sums of ultimately periodic sequences.
  - We write $u\,(v)$ for the infinite sequence
  $$uvvvv\ldots$$
  - Such a sequence w denotes the unique time warp sending any $0 < n < \omega$ to $\Sigma_{i<n} w_i$, where w_i is the ith natural number in w (starting at 0).
  - Presentable time warps are closed by $*$, e.g.,
  $$(2) * (0\,1) = \underbrace{[0\,1]}_{1}\ \underbrace{[0\,1]}_{1}\ \underbrace{[0\,1]}_{1}\ \underbrace{[0\,1]}_{1}\ldots = (1).$$



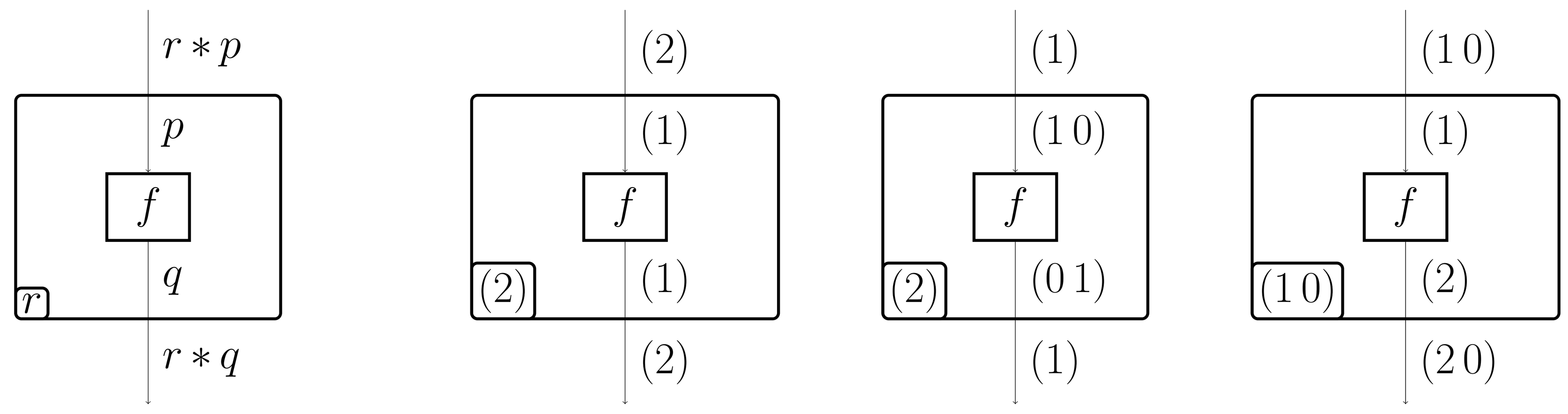
Delays and Local Time Scales

- Assume p and q are time warps such that $q \leq p$. A **delay** allows us to consume at the rate described by q data that was produced at the rate described by p.



Delays can be implemented as finite circular buffers when $p - q$ is bounded.

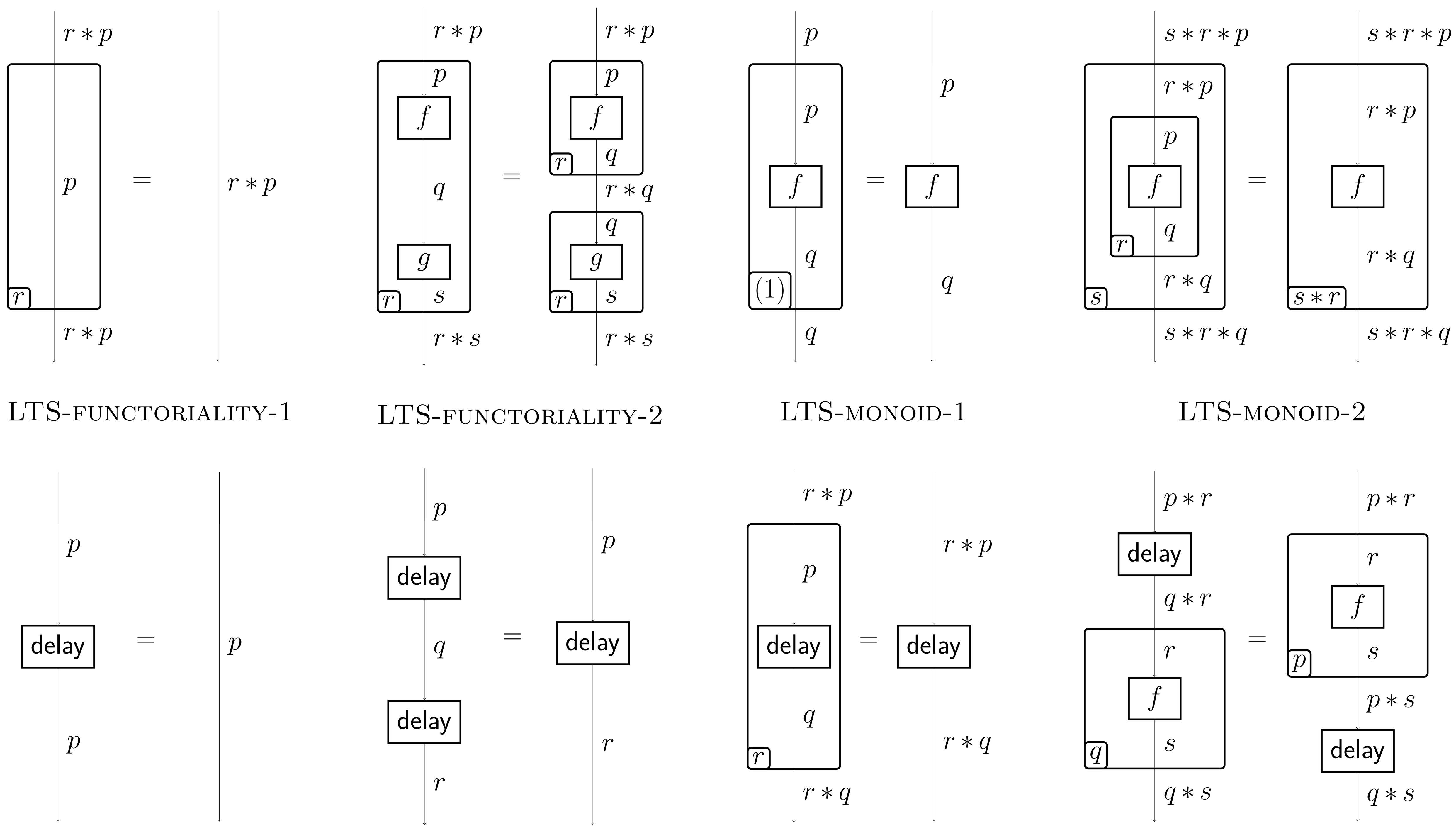
- One may run any block diagram within a **local time scale** driven by a time warp p which controls the relationship between internal and external ticks. For instance, a time scale driven by (2) performs two internal ticks for each external tick.



Local time scales can be implemented as counted loops in the generated code.

Equivalences Between Block Diagrams Involving Delays and Local Time Scales

- The equations on the right show program equivalences involving local time scales and delays. They hold for any blocks $f : p \to q$ and $g : q \to s$.
- They can be used to justify program transformations or to simplify program proof. They also have applications in category theory, where they are the key to *coherence* theorems.
- Each equation belongs to one of three groups.
  1. The first row deals involves only local time scales.
  2. The first half of the second row involves only delays.
  3. The second half of the second row describe the interplay between local time scales and delays.
- We assume that both sides of the equation are well-typed. In LTS-D-2, we assume that $p \geq q$ holds. This entails $r * p \geq r * q$ and $p * r \geq q * r$.
- The equations are justified by a new model of synchronous programming in a certain categorical setting known as the *topos of trees*. It is described briefly in A Generalized Modality for Recursion, LICS'18.
- Many questions are open, such as the completeness of this equational theory for operational equivalence, or the study of associated rewriting systems.



LTS-functoriality-1, LTS-functoriality-2, LTS-monoid-1, LTS-monoid-2

D-functoriality-1, D-functoriality-2, LTS-D-1, LTS-D-2

Application: Optimizing a Slow Block Placed Within a Fast Time Scale